Making Losing Games Pay Off

Shankha Suvra Dam

LIMIT Online Camp (Category C)

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Game A

You are given a biased coin.

- Probability of winning (heads): $\frac{49}{100}$
- Probability of losing (tails): $\frac{51}{100}$

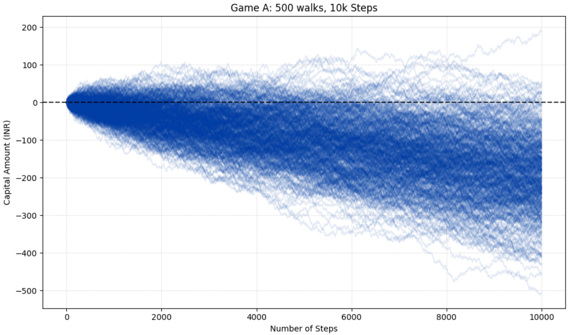
Payout: $+1$ INR for a win, -1 INR for a loss.

Is this a good deal in the long run?

Game A: Analysis

How do you discuss about whether it is a *good deal* in the long run for such situations?

We use *expected result* of the game!



Game A: Analysis

Let's calculate the expected value (E_A) per round:

$$\begin{aligned} E_A &= \mathbb{P}(\text{Win}) \times (+1) + \mathbb{P}(\text{Lose}) \times (-1) \\ &= \left(\frac{49}{100}\right)(1) + \left(\frac{51}{100}\right)(-1) \\ &= -\frac{2}{100} = -0.02 \end{aligned}$$

Since the expected value of the game per round is negative, overall, it is a *losing* game for us in the long run!

Game B

Let X be your current capital. Your coin depends on the value of $X \pmod 3$.

If $X \equiv 0 \pmod 3$,

Toss **Coin 1**.

- Win (heads): $\frac{9}{100}$
- Lose (tails): $\frac{91}{100}$

If $X \not\equiv 0 \pmod 3$,

Toss **Coin 2**.

- Win (heads): $\frac{3}{4}$
- Lose (tails): $\frac{1}{4}$

Payout: $+1$ INR for a win, -1 INR for a loss.

Is this a good deal in the long run?

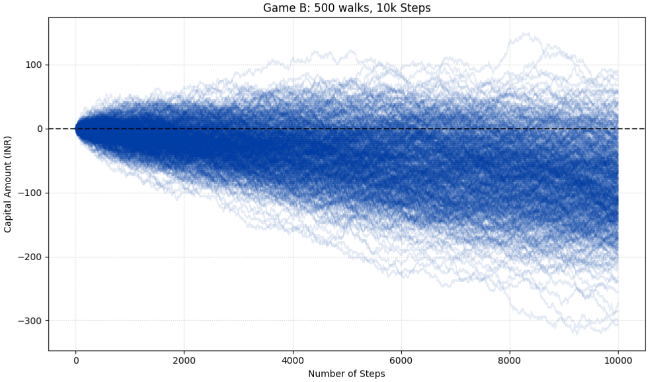
Game B: Analysis

Let's calculate the expected value assuming our capital X is equally likely to be $0, 1$, or $2 \pmod 3$.
We would toss Coin 1 one-third of the time, and Coin 2 two-thirds of the time:

$$\begin{aligned} E_B &= \frac{1}{3}\mathbb{E}[\text{Coin } 1] + \frac{2}{3}\mathbb{E}[\text{Coin } 2] \\ &= \frac{1}{3}\left(-\frac{82}{100}\right) + \frac{2}{3}\left(\frac{1}{2}\right) \\ &= -\frac{82}{300} + \frac{100}{300} = \frac{18}{300} > 0 \end{aligned}$$

A positive expected value! So Game B is a winning game, right?

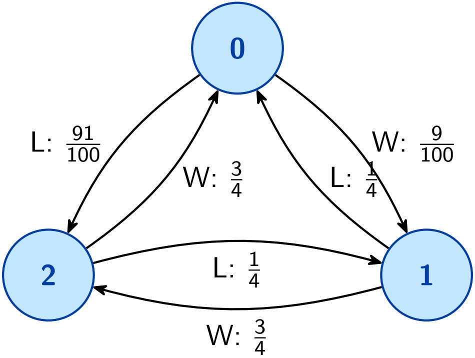
Game B: Analysis



The distribution seems to be skewed negative! Because we can see upper bound to be ≈ 100 and lower bound to be ≈ -325

Game B: Analysis

The mistake is assuming our capital is uniformly distributed. In reality, the game acts as a *Markov Chain*, where outcomes dictate how often we visit each state.



Game B: Analysis

Observation: The probability of being funneled towards *State 0* looks rather high compares to staying at *States 1* and *2*. Hence, assuming uniform distribution over the states is impractical.

Let's formalise this using *transition matrices* and *stationary distributions*.

Definiting Transition Matrix

Definition

Transition Matrix: The transition matrix of a markov chain with n states is defined as

$$T = [p_{ij}]_{1 \leq i,j \leq n}$$

where p_{ij} is defined as the probability of moving from i^{th} state to j^{th} state.

Definiting Transition Matrix

For our scenario, the transition matrix is defined as follows:

$$T = \begin{bmatrix} 0 & \frac{9}{100} & \frac{91}{100} \\ \frac{1}{4} & 0 & \frac{3}{4} \\ \frac{3}{4} & \frac{1}{4} & 0 \end{bmatrix}$$

Defining Stationary Distribution

Definition

Stationary Distribution (π): A row vector $\pi = [\pi_1, \pi_2, \ldots, \pi_n]$ is a stationary distribution for a Markov chain with transition matrix T if it satisfies:

$$\pi T = \pi$$

and the sum of its elements is equal to 1 ($\sum \pi_i = 1$).

This vector represents the true, long-run proportion of time the system spends in each state, regardless of where it started.

Defining Stationary Distribution

We initially start at State 0. Therefore, our **initial distribution** (π_0) is:

$$\pi^{(0)} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

As we play many rounds, the probability of being in each state evolves ($\pi^{(n)} = \pi^{(0)} T^n$) and eventually stabilizes at the stationary distribution $\pi = [\pi_0, \pi_1, \pi_2]$.

Solving the system $\pi T = \pi$ for our specific matrix T, we get:

- $\pi_0 = \frac{325}{843} \approx 38.5\%$ *(Playing Coin 1)*
- $\pi_1 + \pi_2 = \frac{518}{843} \approx 61.5\%$ *(Playing Coin 2)*

Game B: Analysis

Let's calculate the true expected value (E_B) per round using our stationary distribution:

$$\begin{aligned} E_B &= \pi_0 \mathbb{E}[\text{Coin 1}] + (\pi_1 + \pi_2)\mathbb{E}[\text{Coin 2}] \\ &= \left(\frac{325}{843}\right)\left(-\frac{82}{100}\right) + \left(\frac{518}{843}\right)\left(\frac{1}{2}\right) \\ &= -\frac{26650}{84300} + \frac{25900}{84300} \\ &= -\frac{750}{84300} < 0 \end{aligned}$$

Since the true expected value of the game per round is negative, overall, it is a *losing* game for us in the long run!

Game C

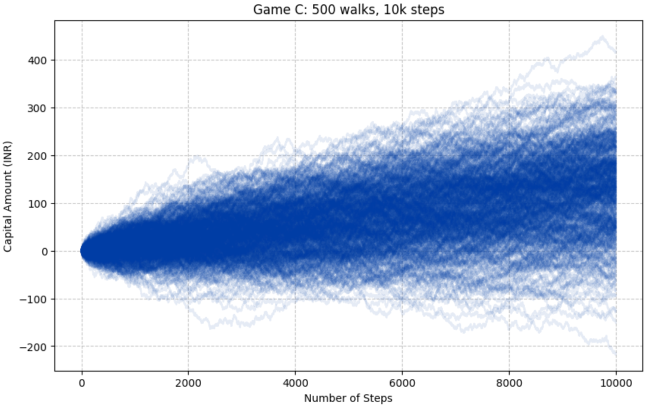
Before every turn, toss a *fair* coin.

- If Heads (50%): Play **Game A**
- If Tails (50%): Play **Game B**

Payout: $+1$ INR for a win, -1 INR for a loss.

Is this a good deal in the long run?

Game C: Computational Results



The distribution seems to be skewed positive! Because we can see upper bound to be ≈ 350 and lower bound to be ≈ -150

Game C: Analysis

Is this just a fluke? Nope!

We can rigorously justify it with slight tweaks to our previous apparata!

Game C: Analysis

If $X \equiv 0 \pmod 3$,

Play Game A or Coin 1.

- Win: $\frac{1}{2}\left(\frac{49}{100}\right) + \frac{1}{2}\left(\frac{9}{100}\right) = \frac{29}{100}$
- Lose: $\frac{71}{100}$

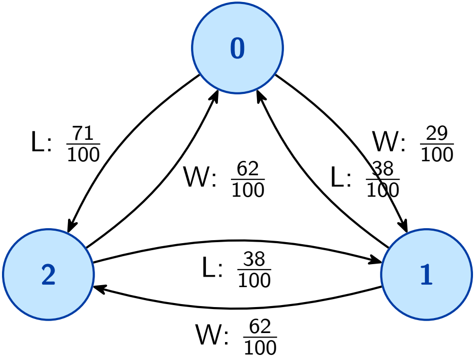
If $X \not\equiv 0 \pmod 3$,

Play Game A or Coin 2.

- Win: $\frac{1}{2}\left(\frac{49}{100}\right) + \frac{1}{2}\left(\frac{75}{100}\right) = \frac{62}{100}$
- Lose: $\frac{38}{100}$

Game C: Analysis

The new probabilities form alter our previous Markov chain slightly.



Game C: Analysis

For Game C, the transition matrix is updated to:

$$T_C = \begin{bmatrix} 0 & \frac{29}{100} & \frac{71}{100} \\ \frac{38}{100} & 0 & \frac{62}{100} \\ \frac{62}{100} & \frac{38}{100} & 0 \end{bmatrix}$$

Solving the system $\pi T_C = \pi$ for our new matrix T_C, we get the stationary distribution for Game C:

- $\pi_0 = \frac{1274}{3690} \approx 34.5\%$ *(Playing at $X \equiv 0$)*
- $\pi_1 + \pi_2 = \frac{2416}{3690} \approx 65.5\%$ *(Playing at $X \not\equiv 0$)*

Game C: Analysis

Let's calculate the true expected value (E_C) per round for the combined game:

$$
\begin{aligned}
E_C &= \pi_0 \mathbb{E}[\text{State } 0] + (\pi_1 + \pi_2)\mathbb{E}[\text{States } 1, 2] \\
&= \left(\frac{1274}{3690}\right)\left(-\frac{42}{100}\right) + \left(\frac{2416}{3690}\right)\left(\frac{24}{100}\right) \\
&= -\frac{53508}{369000} + \frac{57984}{369000} \\
&= \frac{4476}{369000} \approx 0.0121 > 0
\end{aligned}
$$

The expected value is **positive**! Two strictly losing games have combined to create a *winning* game.

Why did this work?!

A couple of observations:

- The win probability in States 1 and 2 dropped from $\frac{75}{100}$ down to $\frac{62}{100}$. We are no longer pushed as hard towards State 0.
- The time spent in State 0 has actively decreased compared to Game B ($38.5\% \to 34.5\%$).

Game B, by itself, was losing primarily due to its structure, trapping us at the bad State 0. Game A adds a random noise that frees us from this issue.

Generalisability of the Paradox

Question: Can we generalise this to two arbitrary processes?

Answer: *Maybe?*

Defining our Sequences

To formalise this phenomenon, we first need to define how we combine two games, Γ_A and Γ_B.

1. Deterministic Sequences

Playing the games according to a fixed, periodic string.

- Example: **[A, B, A, B, . . .]**
- Example: **[A, B, B, A, B, B, . . .]**

2. Probabilistic (Randomized) Sequences

Playing the games by tossing a biased coin before every turn. Let $\gamma \in (0, 1)$ be our mixing parameter.

- Play Γ_A with probability γ.
- Play Γ_B with probability $(1 - \gamma)$.

Parrondo's Paradox: A Rigorous Definition

Let Γ_A and Γ_B be two distinct games that are strictly losing in the long run:

$$\mathbb{E}[\Gamma_A] < 0 \quad \text{and} \quad \mathbb{E}[\Gamma_B] < 0$$

Let Γ_C be a new game constructed by playing Γ_A and Γ_B according to either a **deterministic sequence** or a **probabilistic sequence** with mixing parameter γ.

Parrondo's Paradox occurs if the expected value of the combined game is strictly positive:

$$\mathbb{E}[\Gamma_C] > 0$$

A Deterministic Toy Model

Before we prove things about complex transition matrices, let's look at a much simpler, *deterministic* example to build intuition.

Game α

No matter your capital X, you always lose.

Payout: -1 INR

Game β

Depends on your capital X:

- X is even: Lose 5 INR
- X is odd: Win 3 INR

Game α is trivially losing. Game β is also losing in isolation, as we risk getting continuously trapped on even numbers!

A Deterministic Toy Model

What if we deliberately play the periodic sequence **[**α**,** β**,** α**,** β**, . . .]** starting at $X = 0$?

- **Start:** $X = 0$ *(Even)*
- **Play** α**:** Lose 1. New $X = -1$ *(Odd!)*
- **Play** β**:** Since X is odd, we win 3. New $X = +2$ *(Even)*
- **Play** α**:** Lose 1. New $X = +1$ *(Odd!)*
- **Play** β**:** Since X is odd, we win 3. New $X = +4$ *(Even)*

Every two-game cycle of $[\alpha, \beta]$ guarantees a net profit of $+2$ INR. This is a *winning* game!

The Continuous Mixing Parameter γ

Let's rigorously evaluate the probabilistic sequence using our mixing parameter $\gamma \in (0, 1)$.

If P_A and P_B are the transition matrices for Game A and Game B, the new blended transition matrix for Game C is a *convex combination*:

$$P_C(\gamma) = \gamma P_A + (1 - \gamma) P_B$$

For every choice of γ, there is a corresponding stationary distribution $\pi(\gamma)$ that satisfies our steady-state equation:

$$\pi(\gamma) P_C(\gamma) = \pi(\gamma)$$

The Expected Drift Function

Let $\mathbf{f}_A$ and $\mathbf{f}_B$ be column vectors representing the expected payout for each state in Game A and Game B.
The blended payout vector is $\mathbf{f}_C(\gamma) = \gamma \mathbf{f}_A + (1 - \gamma)\mathbf{f}_B$.

We define the **Expected Drift Function** for the combined game as:

$$E_C(\gamma) = \pi(\gamma) \cdot \mathbf{f}_C(\gamma)$$

To prove a winning combination exists: We must prove that the continuous function $E_C(\gamma)$ has values strictly greater than 0 on the open interval $(0, 1)$.

Can we do this with ANY two games?

Question: Can we achieve this paradoxical result if both Game A and Game B are completely independent of our current capital X?

Theorem: If Γ_A and Γ_B are *state-independent* games with strictly negative expected values, then any combined game Γ_C (deterministic or probabilistic) will also have a strictly negative expected value.

To prove impossibility:

We must prove that for all possible mixing parameters, the expected drift is non-positive:

$$\max_{\gamma \in [0,1]} E_C(\gamma) \leq 0$$

Proof of State Independence Rule

Proof Sketch: Let the expected payoffs of the state-independent games be $\mu_A < 0$ and $\mu_B < 0$.

Assume we play a combined game Γ_C for N total rounds. Let N_A be the number of times we play Γ_A, and N_B be the number of times we play Γ_B, such that $N_A + N_B = N$.

Since the games are state-independent, the expected total payoff after N rounds is strictly linear:

$$\mathbb{E}[\text{Total}] = N_A\mu_A + N_B\mu_B$$

Because $N_A, N_B \geq 0$ (and not both zero), and both $\mu_A, \mu_B < 0$:

$$\mathbb{E}[\text{Total}] < 0$$

Wait, what about Linearity of Expectation?

Doubt: By the linearity of expectation, isn't the expected value of the combined game just the combination of the expected values?

$$\mathbb{E}[\gamma A + (1-\gamma)B] \stackrel{?}{=} \gamma\mathbb{E}[A] + (1-\gamma)\mathbb{E}[B]$$

If this were true, mixing two negative numbers would always yield a negative number. *Why does this fail for Parrondo's games?*

A Linear Dilemma

Linearity of expectation holds for *random variables*, but our game's expectation depends on the **stationary distribution** (π), which fundamentally shifts!

In isolation, Game A and Game B have their own stationary distributions, π_A and π_B.

When we combine the games, the new stationary distribution $\pi(\gamma)$ is **not** a linear combination of π_A and π_B.

$$\pi(\gamma) \neq \gamma\pi_A + (1-\gamma)\pi_B$$

The games are state-dependent. By mixing the transition matrices, we alter the flow of probability between the states, literally changing *how often* we are subjected to the "bad" payouts!

Applications & Limitations

Can we use this to bankrupt a casino?
Sadly, no. Standard casino games (Roulette, Slots, Craps) are rigorously designed to be **state-independent**.

As we proved earlier, no combination of state-independent losing games can ever yield a positive expected value. *The house always wins.*

Then why do mathematicians and scientists care?
Parrondo's Paradox naturally models complex, real-world systems!

- **Physics (Brownian Motors):** Extracting directed, forward motion from purely random thermal noise and asymmetric potentials.
- **Biology (Genetics):** Alternating populations between two

References

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