Logic of Voting: When Math Meets Democracy

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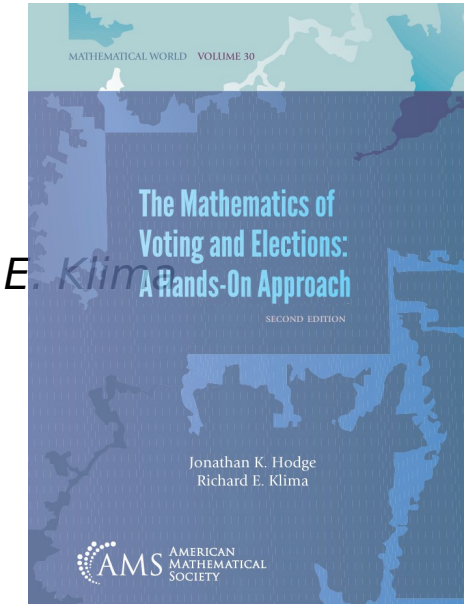
Topics to be Covered

1. What is a Voting System?
2. Examples of Voting Systems
3. A Strange Deadlock
4. Fairness Criteria in Voting Systems
5. Arrow’s Impossibility Theorem
6. Strategy Proofing*

References

beginner friendly



1. *The Mathematics of Voting and Elections: A Hands-on Approach* by *J.K. Hodge, R.E. Klima*
(Volume 30 of *Mathematical World* by *American Mathematical Society*)

2. *Choice Theory – A very short introduction* by *M. Allingham*
Oxford University Press, 2002

3. *Handbook of Computational Social Choice* by *F. Brandt*, *V. Conitzer*, *U. Endriss*, *J. Lang* and *A.D. Procaccia*
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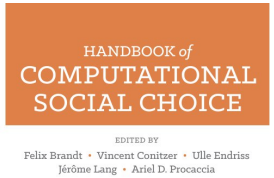
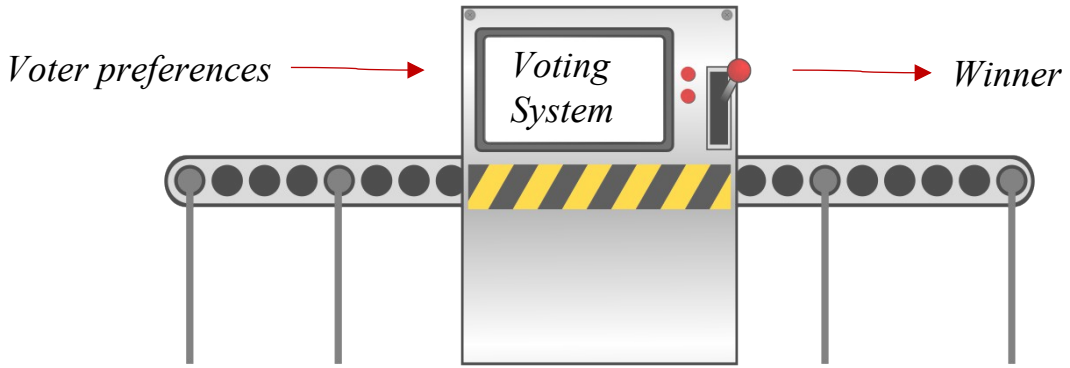


What is a Voting System?

A voting system is a set of rules that determine how the preferences of a population are aggregated to arrive at a winning alternative.

candidates up for election are called alternatives

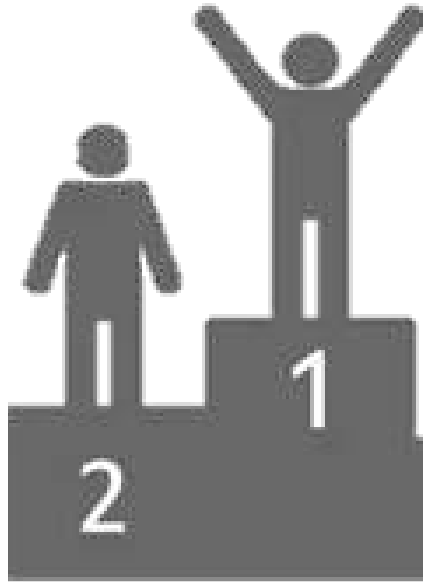
So essentially, a voting system is like a machine, that collects the population's preferences amongst the candidates, and outputs a winning candidate, following certain rules.



Examples of Voting Systems: Majority System

If there are *2* candidates participating in the election, then one of them needs to gain more 50% of the vote to win the election!

This is one of the simplest forms of a voting system, where voters cast one vote to one of the two candidates, and the winner is decided on who got more votes



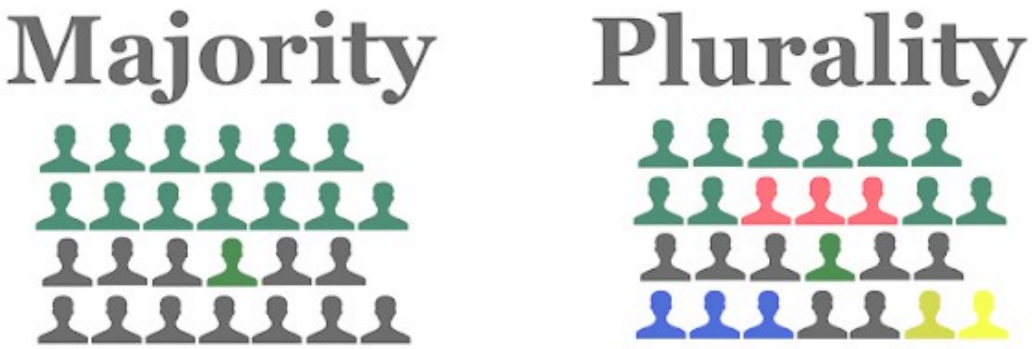
Examples of Voting Systems: Plurality System

What happens when there are more than 2 candidates?

> There are multiple systems that try to tackle this question, since 50% of the vote is not mathematically guaranteed to any candidate!

Plurality system is added as an extension to the Majority system, with all voters having to choose one candidate over the rest, and the winner being the candidate with the most votes (regardless of vote share)

Used in most of the world for elections, including India, United Kingdom, and USA



Examples of Voting Systems: Plurality System

A	B	C	D	E	F	G
15	14	5	13	10	9	6

A major con of this system is that increasing the number of candidates causes people with support from very small fractions of the populace to win.

As can be seen, that A wins, despite winning only 20.83% of the voteshare! (15 out of 72 votes)

This is where Ranked Choice Voting Systems come into play. We will visit them one by one with the following example

A Strange Deadlock

5	4	3	2
Tantri	Kalia	Mapui	Shanti
Suppandi	Shanti	Suppandi	Mapui
Shanti	Suppandi	Shanti	Kalia
Mapui	Mapui	Kalia	Suppandi
Kalia	Tantri	Tantri	Tantri

The citizens of Tinklenagar wish to finally embrace a voting system to let everyone's voices be heard.

In their inaugural mayoral election, we have 5 candidates: **Kalia**, **Mapui**, **Shanti**, **Suppandi** and **Tantri**

What voting system should they choose?

A Strange Deadlock: Plurality

5	4	3	2
Tantri	Kalia	Mapui	Shanti
Suppandi	Shanti	Suppandi	Mapui
Shanti	Suppandi	Shanti	Kalia
Mapui	Mapui	Kalia	Suppandi
Kalia	Tantri	Tantri	Tantri

When we count the number of votes for all the candidates, we find the following result:

Tantri: 5
Kalia: 4
Mapui: 3
Shanti: 2

Therefore, if the elections are conducted by Plurality voting, **Tantri** shall be the winner of the election!

Examples of Voting Systems: Borda Count

Proposed by: Nicholas of Cusa (1435), later popularised by Jean-Charles de Borda (1770)

How it works:
Voters rank all the candidates in order of their preference, and points are assigned per position.
1st preference – *n-1 points*, 2nd preference – *n-2 points*, … nth preference – 0 points

Pros: Whoever scores the maximum number of points wins the election
1. Encourages consensus- A broadly acceptable candidate has higher chances of winning
2. Reduces vote-splitting- Similar candidates do not harm each other's vote shares

A Strange Deadlock: Borda Count

Points	5	4	3	2
4	Tantri	Kalia	Mapui	Shanti
3	Suppandi	Shanti	Suppandi	Mapui
2	Shanti	Suppandi	Shanti	Kalia
1	Mapui	Mapui	Kalia	Suppandi
0	Kalia	Tantri	Tantri	Tantri

Using the Borda system proposed by *Shanti*, we get the following results:

Kalia: (4x4) + 3 + (2x2) = **23**
Mapui: 5 + 4 + (3x4) + (2x3) = **27**
Shanti: (5x2) + (4x3) + (3x2) + (2x4) = **36**
Suppandi: (5x3) + (4x2) + (3x3) + 2 = **34**
Tantri: (5x4) = **20**

Therefore, if the elections are conducted by Borda voting, **Shanti** shall be the winner of the election!

Examples of Voting Systems: Plurality with Runoff

How it works:

The voting is split into two rounds:

1. First Round: The first preferences of voters are considered. If a candidate gets a majority, they win, otherwise we go to the second round with the top two candidates.
2. Second Round: The candidate with greater number of votes after runoff wins

3	**2**	**3**
A	B	C
B	A	B
C	C	A

In the example:

1. **First round**: B is eliminated (2 votes)
2. **Second round:** A wins (5 votes) against C (4 votes)

Used in *France*, *Brazil*, *Indonesia*, *Argentina*

A Strange Deadlock: Plurality with Runoff

5	4	3	2
Tantri	Kalia	Mapui	Shanti
Suppandi	Shanti	Suppandi	Mapui
Shanti	Suppandi	Shanti	Kalia
Mapui	Mapui	Kalia	Suppandi
Kalia	Tantri	Tantri	Tantri

Out of the five candidates, none of them got more than 50% (i.e. 7 votes) of the votes.

The top two candidates in terms of votes therefore are:
Tantri: 5 votes
Kalia: 4 votes

Now, we shall consider the new updated table to determine which of the two wins!

A Strange Deadlock: Plurality with Runoff

5	4	3	2
Tantri	Kalia	~~Mapui~~	~~Shanti~~
~~Suppandi~~	~~Shanti~~	~~Suppandi~~	~~Mapui~~
~~Shanti~~	~~Suppandi~~	~~Shanti~~	Kalia
~~Mapui~~	~~Mapui~~	Kalia	~~Suppandi~~
~~Kalia~~	Tantri	Tantri	Tantri

Going by the voters' preference orders, the votes come out to be:

Tantri: 5 votes
Kalia: 9 votes

Therefore, if the elections are conducted by Plurality with Runoff voting, **Kalia** shall be the winner of the election!

Examples of Voting Systems: Instant Runoff

5	4	3	2
Tantri	Kalia	Mapui	Shanti
Suppandi	Shanti	Suppandi	Mapui
Shanti	Suppandi	Shanti	Kalia
Mapui	Mapui	Kalia	Suppandi
Kalia	Tantri	Tantri	Tantri

How it works:
The first preferences of voters are considered. If a candidate gets a majority, they win, otherwise we go repeat, with the candidate with the lowest number of votes being removed, and the voters' next preference being considered.
Eventually, one winner remains.

Used in *Australia*, *Ireland*

However, it is frequently criticised for non-monotonicity (defined later)

A Strange Deadlock: Instant Runoff

5	4	3	2
Tantri	Kalia	Mapui	Shanti
Suppandi	Shanti	Suppandi	Mapui
Shanti	Suppandi	Shanti	Kalia
Mapui	Mapui	Kalia	Suppandi
Kalia	Tantri	Tantri	Tantri

The tally of the votes when considering the first preferences come out to be:

Tantri: 5 votes
Kalia: 4 votes
Mapui: 3 votes
Shanti: 2 votes
Suppandi: **0 votes**

So, by design, Suppandi is the first one to be eliminated, and the table is thus modified for the next tally.

A Strange Deadlock: Instant Runoff

5	4	3	2
Tantri	Kalia	Mapui	~~Shanti~~
~~Suppandi~~	~~Shanti~~	~~Suppandi~~	Mapui
~~Shanti~~	~~Suppandi~~	~~Shanti~~	Kalia
Mapui	Mapui	Kalia	~~Suppandi~~
Kalia	Tantri	Tantri	Tantri

The tally of the votes when considering the updated preferences come out to be:

Tantri: 5 votes
Kalia: 4 votes
Mapui: 3 votes
Shanti: **2 votes**

Thus Shanti is the next one to be eliminated, leaving the competition tight between Kalia, Mapui and Tantri!

A Strange Deadlock: Instant Runoff

5	4	3	2
Tantri	~~Kalia~~	Mapui	~~Shanti~~
~~Suppandi~~	~~Shanti~~	~~Suppandi~~	Mapui
~~Shanti~~	~~Suppandi~~	~~Shanti~~	~~Kalia~~
Mapui	Mapui	~~Kalia~~	~~Suppandi~~
~~Kalia~~	Tantri	Tantri	Tantri

The tally of the votes when considering the updated preferences come out to be:

Tantri: 5 votes
Kalia: **4 votes**
Mapui: 5 votes

Surprisingly, Kalia, who had been leading ahead of Mapui falls short this time, since all of Shanti's voters decided to back Mapui over Kalia!

The final competition remains between Tantri and Mapui!

A Strange Deadlock: Instant Runoff

5	4	3	2
Tantri	~~Kalia~~	Mapui	~~Shanti~~
~~Suppandi~~	~~Shanti~~	~~Suppandi~~	Mapui
~~Shanti~~	~~Suppandi~~	~~Shanti~~	~~Kalia~~
Mapui	Mapui	~~Kalia~~	~~Suppandi~~
~~Kalia~~	Tantri	Tantri	Tantri

The last round of tallying the votes results in the following:

Tantri: **5 votes**
Mapui: 9 votes

So, Tantri is eliminated!

Therefore, if the elections are conducted by Instant Runoff voting, **Mapui** shall be the winner of the election!

A Strange Deadlock

5	4	3	2
Tantri	Kalia	Mapui	Shanti
Suppandi	Shanti	Suppandi	Mapui
Shanti	Suppandi	Shanti	Kalia
Mapui	Mapui	Kalia	Suppandi
Kalia	Tantri	Tantri	Tantri

Let us recap:

We have considered four different voting systems, and we have gotten four different winners!

Plurality voting: Tantri
Borda: Shanti
Plurality with Runoff: Kalia
Instant Runoff: Mapui

What voting system would be the most fair in this situation?

Arrow's Impossibility Theorem

In 1951, Kenneth Arrow shook the world with his book "*Social Choice and Individual Values*", in which while trying to ascertain the fairness of the various *ranked* voting systems after facing the same dilemma as us.
He introduced five notions of fairness:

1. Universality
2. Positive Association of Social and Individual Values (Monotonicity)
3. Independence of Irrelevant Alternatives (IIA)
4. Citizen Sovereignty
5. Non-Dictatorship

After which he stated his impossibility theorem.

Arrow's Impossibility Theorem

"*For an election with more than two candidates, it is impossible for a ranked voting system to satisfy all five notions of fairness*"

To be able to comprehend the implications of this theorem, we will first need to discuss the notions of fairness!
But before that, let us talk about some basic terminology

$A \succ B \succ C$ represents a preference order, where C is ranked below B and B below A

Individual Preference Order is defined as the preference order of an individual voter

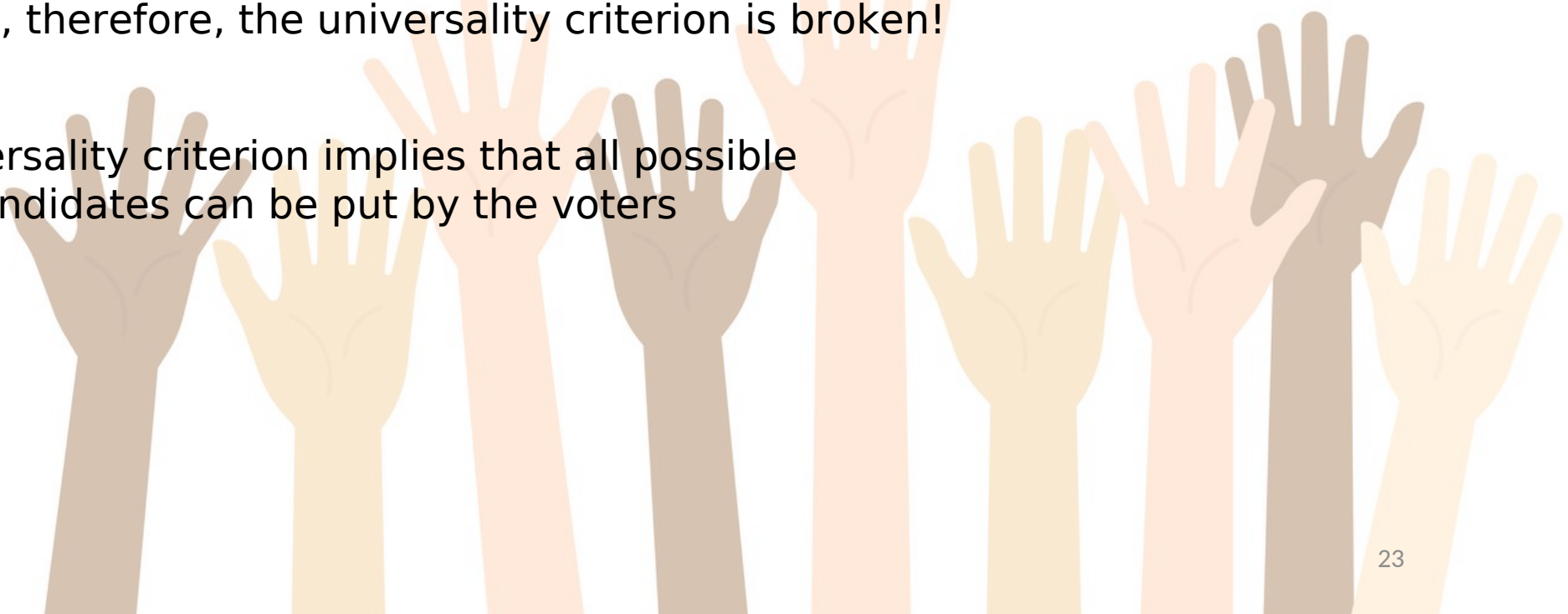
Societal Preference Order is defined as the preference order created as a result of the voting system in question

Arrow’s Impossibility Theorem: Universality

Universality criterion necessitates the voting system to be able to process any possible set of individual preferences amongst the available alternatives

Example: Say in the previously discussed example of A Strange Deadlock, in the voting system, the voters are not allowed to put Tantri below Mapui, therefore, the universality criterion is broken!

Conclusion: Universality criterion implies that all possible permutations of candidates can be put by the voters



Arrow's Impossibility Theorem: Monotone

A voting system is called **monotone**, if changes favourable only to a particular candidate in individual preference orders **cannot** cause that candidate to be ranked lower in the resulting societal preference order

Therefore, the condition of monotonicity basically requires the voting system to be monotone.

Example: So, say there are three candidates, A, B and C, in an election, with the given preference orders of the voters. Now, let there is a change of heart in one of the voters of first column which causes them to rank A above C instead in their ballot.

If the voting system punishes candidate A for having a better showing than previous by ranking them below in the societal preference order, then our voting system is NOT monotone.

Personal Preference Orders		
5	**4**	**1**
C	A	B
A	B	A
B	C	C

Initial societal preference order: $A \succ C \succ B$

Arrow's Impossibility Theorem: IIA

Independence of Irrelevant Alternatives (IIA) necessitates that:
*If a group of voters prefers candidate A over candidate B, then by introducing a third, unrelated candidate C should **not** change the outcome between A & B*

Interestingly, plurality voting does **not** follow IIA. Let us see this with an example!

45	**40**	**15**
Tantri	Mapui	Shanti
Mapui	Shanti	Mapui
Shanti	Tantri	Tantri

If we consider the preference of the electorate between Tantri and Mapui, it's clear that they prefer Mapui (55 votes) over Tantri (45 votes)

Had the election been done between just the two of them, Mapui would've won, but due to Shanti splitting the vote, Tantri won instead!

Arrow's Impossibility Theorem: Citizen Sovereignty

The criterion of Citizen Sovereignty states that:
There should never be a pair of candidates, say A and B, such that A is always preferred over or tied with B in the societal preference order ***irrespective*** *of how the voters vote*

Examples: Say, in our initial example of A Strange Deadlock, Tantri always won the elections, regardless of the votes cast. That'd be a breach of Citizen Sovereignty

Accidental Breach of Citizen Sovereignty: In some cases, citizen sovereignty may be breached due to dodgy implementations of the voting mechanism (for example EVM), causing the voter's vote to not be recorded correctly. To prevent this in India, the ability of confirming the vote is given to the voter via VVPAT machines, which print a small slip with the name and symbol of the candidate voted for

Arrow’s Impossibility Theorem: Dictatorship

The criterion for Non-Dictatorship states that:
There should never be a particular voter (a dictator), such that for any pair of candidates A and B, if the dictator prefers A over B, then society will prefer A over B

Example: The most trivial example of this is a system where regardless of how everyone else votes, there exists only one person whose preference order shall be considered in the system
A very interesting case of dictatorship can be seen in resource allocation problems! In such problems, candidates submit their preference orders, and then serially, are able to pick their top-choice. So essentially, each person acts as a dictator over the remaining options in the turn.
An example of such in real life would be JoSAA counselling, done for allocating seats in colleges via JEE scores.

Arrow’s Impossibility Theorem

Seeing the five criteria for fairness in a voting system, all of the requirements sound very reasonable. However, Kenneth Arrow proved that all these five criteria cannot be satisfied at once in a voting system, which was an astonishing result for the time.

The proof for this theorem, while elementary, is too long for the time constraints

Reference (1) wonderfully motivates the proof of this theorem in their chapters 4 & 5

Arrow's Impossibility Theorem

Voting System	Universality	Monotone	IIA	Citizen Sovereignty	Non-Dictatorship
Plurality	Yes	Yes	No	Yes	Yes
Borda Count	Yes	Yes	No	Yes	Yes
Plurality with Runoff	Yes	Yes	No	Yes	Yes
Instant Runoff	Yes	No	No	Yes	Yes
Dictatorship	Yes	Yes	Yes	Yes	No

Putting our voting systems up to test, we find that all of our democratic voting systems seem to fail IIA!

In actuality, while IIA is a very elegant requirement, it is a very brittle criterion to work with. One will be hard-pressed to find a voting system satisfying IIA.

Arrow's Impossibility Theorem

If IIA is such an impossible condition to satisfy, what do we do then?

1. *Relax IIA:* When considering voting systems, some prefer to use LIIA (Local Independence of Independent Alternatives) to judge if a voting system is fit for them.

2. *Approval Voting:* Approval voting system asks the voters to provide a subset of candidates that they approve of, and the candidate with the most approvals wins the election. While it does NOT satisfy IIA, it is less sensitive to it.

3. *Weighted Voting:* In this family of systems, voters are not treated equally. Every vote has varying influence. While not helpful, it can make IIA less relevant. Used in *IMF, EU Council*

Arrow's Impossibility Theorem: Conclusion

What is the best voting system?

Turns out, there is no *best* voting system. Every system comes with its tradeoffs.

What matters at the end of the day is what trade-offs one is willing to make to consider.

What's next?

K. Arrow's work started major discourse, and revolutionised the field of Social Choice Theory

A major topic in this field is *Strategy-Proofing* where we try to design voting systems in a way to restrict the possibility of gaming the system make their preferences win.

Further, the logic of voting systems has been extended to topics such as Coalition Formation (individuals form alliances to achieve collective goals) and Tournament Solutions (methods of selecting winners in pairwise majority comparisons, essentially finding a winner of a tournament graph)

Questions and Answers

Strategy Proofing

Strategy Proofing refers to designing a voting system where voters have no incentive to misrepresent their preferences

So, the intention is to design rules to encourage honesty, instead of engaging in tricks.

A strategy-proof system removes the ability to gain an advantage by voting insincerely.

Gibbard–Satterthwaite Theorem:
If a voting system:

1. Always provides a single winner (resolute)
2. Allows atleast three possible outcomes
3. Is Strategy Proof

Then the voting system is a dictatorship

A curious example: Random Dictatorship!